A tribute to Euler. Title: Ten styles for product integrals and product differentiation.

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Clarification of the notation.
Throughout this little article we will use the notation $\prod_i^n q_i$ for the discrete product of numbers q_i and the same symbol $\prod$ is used for the product integral $\prod_a^b f(x)^{dx}$.
End of the clarification.

Introduction.
Product integration is the same as ordinary integration instead of adding it all up we are now multiplying everything. Product integration is already over one hundred years old but weirdly enough it does not draw much attention.
The reasons for having so little attention placed on this beautiful corner of mathematics is plain simple:
In my view most professional mathematical people are in fact full blown idiots. Let me illustrate that with a simple example:

Example from the library: In the 1980-ties from the previous century I was a student here in the city of Groningen and in the university library was a booklet about product integration.
It was a relatively old book and the entire book looked pristine except for two pages at the end of the book.
So I wondered why these two pages were so dirty and worn out; I did read the title of those two pages and the title said 'Linearisation'.
So the entire book was pristine while the pages about linearisation were worn out.
That proves that in those years the professional math people in my university were relatively dumb because exponential processes of do not allow for broad linear approximations...

End example.

In this piece we will look at the diverse ways of defining and introducing product integration but also, what has been skipped for the entire century; we will look at the way of differentiation that comes along with product integration.

We will look at 10 more or less different styles to introduce the idea of product integration; with product integration you do the same as with ordinary integrals like $\int f(x)\,dx$ but instead of adding it all up you multiply the stuff.
This will be done in ten styles, the ten styles are:

- Differential equation style.
- Peano style.
- Reinko style.
- Riemann style.
- Leibniz style.
- Anti-primitive style.
- P-function style.
- Step function style.
- Lebesgue style.
- Cauchy style.

A very elementary definition is given by an important class of differential equations. This is the **differential equation style** The solutions to these differential equations are basically the product integrals although no booklet about differential equations says so.
Let $g(x)$ be a smooth positive valued function on the real line $\mathbb{R}$ and look at the next differential equation in the unknown function $f(x)$:

$$f'(x) = g(x) \cdot f(x)$$

For uniqueness we need a boundary condition, for example $f(0) = 1$. In that case the solution is given by

$$f(x) = e^{G(x)} \text{ where } G(x) = \int_0^x g(s)ds \text{ or } G'(x) = g(x)$$

All functions with an integral in the exponent can basically be viewed as being a product integral.

Not only the sheer incompetence of what supposedly should be 'professional mathematicians', one of the oldest definitions of product integration already ensured it had to have a difficult start.
End of this introduction.

Peano series; a difficult start.
The way Peano defined the product integral was not an example of transparency; the product integral of a function $Q(y)$ was defined as next:

$$M(y,0) = 1 + \int_0^y Q(x)dx + \int_0^y \int_0^x Q(x)Q(x_1)dxdx_1 + \int_0^y \int_0^x \int_0^{x_1} Q(x)Q(x_1)Q(x_2)dxdx_1dx_2 + \dots$$

Needless to say: you do not make much friends with definitions like this. But hey, if you differentiate the M to the variable y you see that

$$\frac{\partial M}{\partial y} = Q(y) \cdot M$$

So now you see why this definition gives rise to a horrible start in life: You have to understand what a product integral is to know it satisfies a particular differential equation and after that you can check that the **Peano style** of product integration is indeed some way of defining the thing.

Of course it was a disaster.

Defining Product integrals Reinko style.
In my first year at the local university I wondered why there are additive integrals and why there is not

something like that with multiplication.
Now I knew a fundamental property of the logarithm because from before we had computers we had to multiply everything by hand and that was a whole lot of work. Therefore people used so called logarithm tables together with the property of the logarithm that

$$\log ab = \log a + \log b$$

In words: the logarithm of a product is the sum of the logarithms. That was the key because if a product integral did exist it just had to have this property

$$\log \prod \ ? \ = \int \log \ ?$$

In words: the logarithm of a product integral is the additive integral of the log of the stuff.
Next step: what is supposed to be on the question mark? Now my idea was that you had to raise a function $f(x)$ to the power dx because of another fundamental property of the logarithm:

$$\log\left(a^b\right) = b \cdot \log(a)$$

And almost three decades ago that was the **Reinko style** definition of the product integral:

$$\log \prod f(x)^{dx} = \int \log\left(f(x)^{dx}\right) = \int \log f(x) \cdot dx$$

Defining Product integrals Riemann style.
The main theorem of calculus says that under sufficient conditions we have

$$\int_a^b f(x)dx = F(b) - F(a) \ \text{ or } \ f(b) = f(a) + \int_a^b f'(x)dx$$

This can be proven easily if you use so called Riemann sums; first you make a partition of the closed interval $[a, b] \subset \mathbb{R}$ into n subintervals.
For example the partition with a constant mesh

$$x_i = a + \frac{i}{n} \cdot (b - a) \text{ where } 0 \leq i \leq n$$

works perfect in most cases.
For any given primitive function $F(x)$ you can now define $\triangle F_i := F(x_i) - F(x_{i-1})$ for $0 < i \leq n$ and that gives a telescoping sum

$$\sum_{i=1}^{n} \triangle F_i = F(b) - F(a)$$

After that you define $\triangle x_i := x_i - x_{i-1}$ and you multiply each term of the telescoping sum with 1 as next

$$\sum_{i=1}^{n} \triangle F_i \cdot \frac{\triangle x_i}{\triangle x_i} = \sum_{i=1}^{n} \frac{\triangle F_i}{\triangle x_i} \cdot \triangle x_i = F(b) - F(a)$$

If you now take the limit for $n \to \infty$ this becomes a **Riemann sum** where

$$\lim_{n\to\infty} \frac{\triangle F_i}{\triangle x_i} = f(x) \text{ and } \lim_{n\to\infty} \triangle x_i = dx$$

so that

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\triangle F_i}{\triangle x_i} \cdot \triangle x_i = \int_a^b f(x)dx = F(b) - F(a)$$

Well there is nothing new in that, but beside Riemann sums there are also Riemann products.
Basically you now do not use subtraction like in $\triangle x_i := x_{i+1} - x_i$ but division. Let's denote this as next

$$\Box F_i := \frac{F(x_i)}{F(x_{i-1})}$$

Instead of telescoping sums you now have telescoping products

$$\prod_{i=1}^{n} \Box F_i = \frac{F(b)}{F(a)}$$

And instead of multiplying the terms by 1 we now raise the factors of the telescoping product by 1 via

$$\prod_{i=1}^{n} \Box F_i^{\triangle x_i/\triangle x_i} = \frac{F(b)}{F(a)}$$

Just like the stuff with Riemann sums we now state that

$$\prod_{i=1}^{n} \Box F_i^{\triangle x_i/\triangle x_i} = \left(\prod_{i=1}^{n} \Box F_i^{1/\triangle x_i}\right)^{\triangle x_i} = \frac{F(b)}{F(a)}$$

And again with taking the limit of $n \to \infty$ you will craft the product integral **Riemann style**.

How would a definition **Leibniz style** look like? Well Leibniz is rumoured to be the first to use the d in stuff like $df = f'dx$ or equivalent

$$\frac{df}{dx} = f'(x)$$

In our present century we say that df is a one-form but back in time they viewed df and dx as infinitesimal numbers. Well Leibniz would simply state that e^{dx} and e^{df} are multiplicative infinitesimals. (In modern language multiplicative one-forms.)
And since product of exponentials always give rise to addition in the exponent we must have

$$\prod_a^b e^{df} = e^{\int_a^b df} = e^{f(b)-f(a)}$$

That is all there is to the **Leibniz style** because the Leibniz notation is highly efficient so the math is short.

We proceed with a definition **anti-primitive style**.
The primitive $F(x)$ of a function $f(x)$ is often called the anti-derivative.
Similar you can say that the derivative is the anti-primitive and we are going to find the way we differentiate functions that is the inverse of taking a product integral.

In the previous update named 'tribute to Gauss' we looked at some alternative form of differentiation. And

$$f^*(x) = \lim_{h\to 0} \left(\frac{f(x+h)}{f(x)} \right)^{1/h}$$

was the alternative as presented.
This way of differentiation and the standard product integral are inverse operations just like $f(x)$ and $f'(x)$ are in stating that $f(x) = f(a) + \int_a^x f'(t)dt$.
For positive valued functions on the real line $\mathbb{R}$ and $f^*(x)$ as in the above definition we have the product integral

$$f(x) = f(a) \cdot \prod_a^x f^*(t)^{dt} := f(a) \cdot \exp\left(\int_a^x f'(t)/f(t)\ dt \right)$$

It is obvious that $\int_a^x f'(t)/f(t)\ dt = \log f(x) - \log f(a)$ so this product integral is indeed the way to calculate $f(x)$ given $f^*(x)$.

Ok, given our differentiation $f^*(x)$ we have to make it plausible that $f^*(x) = e^{f'(x)/f(x)}$. But that is amazingly

simple and fundamental; in calculus when you have a continuous function like the exponential map, the limit of the function is the function of the limit.
What do I mean in detail with that?
Now let x_n be a Cauchy sequence of real numbers, that means the sequence has a limit, let's denote that by x

$$\lim_{n\to\infty} x_n = x$$

The continuity property says that if a function g is continuous at x we have

$$\lim_{n\to\infty} g(x_n) = g\left(\lim_{n\to\infty} x_n\right) = g(x)$$

In high school words: You can simply fill in the limiting value. So the calculation for $f^*(x)$ becomes very simple if we use the identity $a = e^{\log a}$.

$$\begin{aligned} f^*(x) = \lim_{h\to 0}\left(\tfrac{f(x+h)}{f(x)}\right)^{1/h} &= \exp\left(\lim_{h\to 0}\tfrac{\log f(x+h)-\log f(x)}{h}\right) \\ &= \exp\left(\tfrac{f'(x)}{f(x)}\right) \end{aligned}$$

Example: Let f be a Gaussian distribution $f(x) = e^{-x^2/2}$, in that case f^* is given by $f^*(x) = e^{-x}$. And the second derivative this way would be $f^{**}(x) = e^{-1}$.

A big handful of possible definitions is given by the product integral **P−function style**.
If you dive into the literature that is out there on the subject of product integration you will find definitions that look very strange and you wonder are we talking about the same stuff yes or no?
What to think of definitions that look like

$$\prod_a^b (1 + df) \quad \text{or} \quad \prod_a^b (1 - df)^{-1}$$

Is that the same object as $\prod_a^b e^{df}$?

The answer is yes, this lies in the realm of the so called $P-$functions. What is a $P-$function?
The official definition works with and open environment of 0 in $\mathbb{C}$ and the $P-$function

is assumed to be analytic in that open environment.
Furthermore the value in $z = 0$ must be 1, just like
the first derivate. So:
1) $P(0) = 1$ and
2) $P'(0) = 1$.

And according to the known official theory, since they are analytic, they allow for a power expansion like

$$P(z) = 1 + z + \sum_{n=2}^{\infty} p_n z^n$$

on that open environment of zero in $\mathbb{C}$.
Easy examples are of course $P(z) = e^z, P(z) = 1 + z$ and the geometric series $P(z) = (1 - z)^{-1}$ but now you know the definition also weird functions like $P(z) = \cos z + \sin z$ are allowed. Furthermore in $\mathbb{C}$ you can rescale any analytic function so that it is a $P-$function.

On other parts of this website there is plenty of stuff about the higher dimensional complex numbers, indeed there is absolutely no reason to restrict the $P-$functions to $\mathbb{C}$ only.
So a wider definition would be given by:
Let $U \subset \mathbb{R}^n$ be an open subset around $X = 0$ and let P be an analytic function obeying the generalized Cauchy-Riemann equations for the particular dimension at hand and let $P(0) = 1$ and $P'(0) = 1$.
If that is fulfilled, this P can also be used to define product integration.

In terms of multiplicative differential one-forms you can say that $e^{df}, 1 + df$ and $(1 - df)^{-1}$ are multiplicative one-forms. But that gives weird stuff at the same time, it is utterly clear that the next differential one-form

$$(1 + df) - 1 = df$$

is an additive one-form. And also $e^{df} - 1$ has to be an additive one-form, that means the next expression looks crazy but it is not untrue

$$\int_a^b (e^{df} - 1) = f(b) - f(a)$$

A very important property of the mathematical sciences
is the fact it has all that **internal coherence**.
Informally said, it does not matter what method you use
in calculating your result. If you make no errors, the
results have to be the same.
For example we use two very different methods:
1) The ancient way of viewing df as an infinitely small numbers, or
2) The df is a modern additive one-form ready for integration.

It does not matter what method you use, always you should find results like

$$\prod_a^b (1 + df) = \prod_a^b (1 + df + (df)^2)$$

Simply said: **the higher powers can be neglected**.

Example of the coherence inside math:
Most people have seen the Euler thing that says

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$$

Now we know the $P-$function stuff, we also have

$$\lim_{n\to\infty} \left(1 + \frac{1}{n} + \left(\frac{1}{n}\right)^2\right)^n = e$$

Step function style. Often integration is defined via
a limiting process with step functions. Step functions, like
the name says it so, are constant on intervals and you can
use step functions as an approximation of a continuous
function $f(x)$ from below and from above.
If the lower and upper limit of finer and finer step functions
converge to the same value, in that case we say that $f(x)$ is
integrable.
Suppose we have a constant function $s(x)$ on some interval $[a, b] \subset \mathbb{R}$.
It is trivial that

$$\int_a^b s(x)\, dx = \int_a^b c\, dx = c(b - a).$$

For a product integral this gives

$$\prod_a^b s(x)^{dx} = \prod_a^b c^{dx} = \exp\left(\int_a^b \log c\, dx\right) = c^{b-a}$$

An important property of product integrals is that if you split the interval of integration you must multiply the subsets in order to get the product integral over the entire interval. I mean

$$\forall\, b, a < b < c : \prod_a^c f(x)^{dx} = \prod_a^b f(x)^{dx} \cdot \prod_b^c f(x)^{dx}$$

So if a step function has n different values on a given interval, after product integration you get n numbers that you must multiply.
With using step functions we still assume these step functions are continuous except where they make the jumps.
Integration of non-continuous functions is done with the help of Lebesgue integration theory and that is our next style:

Product integration **Henri Lebesgue style**.
For myself speaking I never use Lebesgue integration but since it is in the standard mathematical toolbox why not waste some words on it?
Henri Lebesgue introduced concepts like measure-ability of subsets and used so called simple functions that converge towards the function you would like to take your integral on.
Normally if we are integrating stuff like $\int_a^b f(x)\, dx$ we more or less scan the $x-$axis and just add up everything we encounter.
Lebesgue integration turns it around: You scan the $y-$axis and for every value you find with your simple functions you multiply this by the measure that belongs to every found value.
For example a simple function $s(x)$ has the next values:
$s(x) = 1$ on the half open interval $[0, 1>$ and
$s(x) = 2$ on the half open interval $[1, 5>$ and
$s(x) = 0$ everywhere else.
The measure of both intervals equals 1 and 4, now the Lebesgue interval is defined as

$$\begin{aligned} \textstyle\int_{[0,5>} s\, d\mu \quad &= 1 \cdot \mu([0, 1>) + 2 \cdot \mu([1, 5> 0 \\ &= 1 \cdot 1 + 2 \cdot 4 = 9. \end{aligned}$$

At first sight this does not look like much of a deal; this looks just like ordinary Riemann integration written down in a difficult way.
But there is more to Lebesgue integration than the 'first glimpse'. For example look at the next function that we define for $x \in [0, 1]$:
$f(x) = \infty$ if x is a fraction (or if $x \in \mathbb{Q}$),

$f(x) = 1$ if x is irrational (or if $x \in \mathbb{R} \setminus \mathbb{Q}$).
Now using telescoping sums or whatever what does not bring a good answer but the Lebesgue way of integration simply remarks the value 1 has measure 1 while the places where it is infinity has a measure of 0.
So the integral can be done

$$\int_{[0,1]} f \, d\mu = 1$$

A definition of product integration Lebesgue style would look like this

$$\prod_{X \subset \mathbb{R}} f^{d\mu}$$

Of course you must view this as a limiting process using simple functions that approximate the stuff you want to integrate.

Finally we arrive at the **Cauchy style** of product integration and this little part is my tribute to Euler.
Recall that above one of the ways for crafting a way to product integration was using a derivative that was defined as

$$f^*(x) := \lim_{h \to 0} \left(\frac{f(x+h)}{f(x)} \right)^{1/h}$$

And in terms of ordinary differentiation this equals

$$f^*(x) = \exp\left(\frac{f'(x)}{f(x)} \right)$$

Although this definition is clearly related to product integration, there is still addition used in this definition.
What would you get if you tried to get that addition out and craft a new way of differentiation that uses only multiplication (thus also division) and raising stuff to some power?
As a first thought something like

$$\lim_{h \to 1} \left(\frac{f(x \cdot h)}{f(x)} \right)^{?}$$

comes to mind.
For me it is a bit hard to explain to you why on the question mark there has to be $1/\log h$, that is because the 1 over $\log h$ comes from a branch that I name 'stripe and lightning theory' and stripe and lightning theory is so outlandish that I better not show it to you because you will instantly think I am utterly and totally crazy...

Ok, let's try this definition.
The new way of differentiating now looks like

$$\lim_{h \to 1} \left(\frac{f(x \cdot h)}{f(x)} \right)^{1/\log h}$$

We must apply a symbol to it like $f'(x)$ or $f^*(x)$, so we just choose a symbol, let's say $f^\circ(x)$.

$$f^\circ(x) := \lim_{h \to 1} \left(\frac{f(x \cdot h)}{f(x)} \right)^{1/\log h}$$

The limit might look a little bit discouraging but with the help of a charlatan named 'l Hopital (and his rule; the rule of 'l Hopital) it is not that hard to crack

$$f^\circ(x) := \lim_{h \to 1} \left(\frac{f(x \cdot h}{f(x)} \right)^{1/\log h} = \lim_{h \to 1} \exp \left(\frac{\log f(x \cdot h) - \log f(x)}{\log h} \right) \stackrel{1}{=} \lim_{h \to 1} \exp \left(\frac{xh \cdot f'(x)}{f(x)} \right) \stackrel{2}{=} \exp \left(\frac{x \cdot f'(x)}{f(x)} \right) \tag{1}$$

Explanation for the equivalence transitions $\stackrel{1}{=}$ and $\stackrel{2}{=}$ goes a next:
$\stackrel{1}{=}$ we observe a limit like $0/0$ because $h \to 1$ so we apply the rule of the charlatan 'l Hopital and differentiate numerator and denominator with respect to the variable h.
$\stackrel{2}{=}$ Because of continuity we can simply fill in $h = 1$ in order to get the limit.

This all looks very difficult but there is a nice collection of monkeys that will come out of the sleeve:
We are going to calculate $f^\circ(x)$ for the basic building blocks of polynomials, so for $f(x) = x^n$.
Recall that the ordinary (additive) derivative equals $f'(x) = n \cdot x^{n-1}$ therefore

$$f^\circ(x) = \exp \left(\frac{x \cdot f'(x)}{f(x)} \right) = \exp \left(\frac{x \cdot n \cdot x^{n-1})}{x^n} \right) = e^n$$

The result is so cute, let's make a theorem out of it:

Theorem of the Cousin of the Great Tuthola:
For all $n \in \mathbb{N}$ let f be the standard polynomial $f(x) = x^n$.
The **Cauchy style** of differentiating gives the result

$$f^\circ(x) = e^n$$

Proof: See the calculations and ideas above.

Some folks might wonder as why this should be named Cauchy style. Indeed if you look at the way of differentiation this name is not utterly clear.
So let's go the path of anti-differentiation and look at the way the product integral looks using this way of differentiation.
Again we need a new symbol because this way of product integration is not the same as we studied above.
Decades ago I used the wedge symbol, so stuff would look like

$$\wedge_a^b f^\circ(x)^{dx}$$

all is very interesting, but how do we relate this wedge stuff to all those other definitions of the product integral?
Now since in the way we take the **Cauchy differentiation** we observe an extra multiplication with x.
So we compensate for that with division by x.
Therefore if we express the wedge in terms of the standard product integral $\prod_a^b$ we get

$$\wedge_a^b f(x)^{dx} = \prod_a^b f(x)^{dx/x}$$

Or, in terms of the two ways of multiplicative differentiation so in terms of $f^*(x)$ and $f^\circ(x)$

$$\wedge_a^b f^\circ(x)^{dx} = \prod_a^b f^*(x)^{x\,dx}$$

Oh oh, if I look at my own work in the last lines I am pretty sure not much people will understand it.
So shall I craft myself a 10-thousand-Jahriches-Reich in order to catch a bit more understanding people...
Or shall I not do that?

No I will not craft a 10 thousand year long empire.

Ok, it is about time we calculate a few of those product integrals and I would like to start with an improper one.
With additive integrals the values $\pm\infty$ have to be taken with a bit of care, whether it is the value of the function or if the domain of integration is infinitely large.

We are going to calculate the product integral of the sinus, with product integration the value 0 gives also improper integrals. Therefore we are going to integrate over the half open interval $< 0, \pi/2]$.
So we want to know what

$$\prod_0^{\pi/2} \sin x^{\ dx} \text{ is.}$$

Remark that officially you must view this as a limiting process

$$\lim_{a \downarrow 0} \prod_a^{\pi/2} \sin x^{\ dx}$$

So we want to know the additive integral of $\log \sin x$ because

$$\prod_0^{\pi/2} \sin x^{\ dx} \ = \ \exp\left(\int_0^{\pi/2} \log \sin x \ dx\right)$$

Without actually calculating this additive integral we simply use the fact that

$$\int_0^{\pi/2} \log \sin x \ dx = -\log 2 \cdot \frac{\pi}{2}$$

So that

$$\prod_0^{\pi/2} \sin x^{\ dx} = e^{-\log 2 \cdot \pi/2} = \sqrt{2^{-\pi}}$$

A beautiful evaluation of the additive integral can for example be found in a book with the title **Integral Calculus** (New age publishers) written by **H.S. Dhami**. Look at page 28 and 29. But the additive integral $\int_0^{\pi/2} \log \sin x \ dx$ is also a result in Fourier analysis.

In order to appreciate a little bit what we are actually doing it is not unwise to view everything as the limit of an infinite product.
We split the half open interval $< 0, \pi/2]$ into n parts of equal size. All subintervals of the partition have width

$$\triangle_n = \frac{\pi/2}{n}$$

and the value of $\sin x$ in each subinterval is evaluated at the right side of the subinterval because we must avoid the zero into our infinite product. Stuff would look like

$$\prod_{i=1}^{n} (\sin i\triangle_n)^{\triangle_n} = \sqrt[n]{\left(\sin(\frac{\pi/2}{n}) \cdot \sin(\frac{2\pi/2}{n}) \cdots \sin(\pi/2)\right)^{\pi/2}}$$

It is elementary stuff that for a positive number a and $b, c \in \mathbb{R}$
we have $(a^b)^c = a^{bc} = (a^c)^b$. That means we can bring the
power $\pi/2$ outside the $\sqrt[n]{}$

$$\sqrt[n]{\left(\sin(\frac{\pi/2}{n}) \cdot \sin(\frac{2\pi/2}{n}) \cdots \sin(\pi/2)\right)^{\pi/2}} = \left(\sqrt[n]{\sin(\frac{\pi/2}{n}) \cdot \sin(\frac{2\pi/2}{n}) \cdots \sin(\pi/2)}\right)^{\pi/2}$$

But we already know that if we take it to the limit $n \to \infty$
that the outcome equals $\sqrt{2^{-\pi}} = (1/2)^{\pi/2}$.
Therefore we must have

$$\lim_{n\to\infty} \sqrt[n]{\sin(\frac{\pi/2}{n}) \cdot \sin(\frac{2\pi/2}{n}) \cdot \sin(\frac{3\pi/2}{n}) \cdots \sin(\pi/2)} = \frac{1}{2}$$

And by all standards this is a surprising result:
From the viewpoint of multiplicative averages (I mean the
geometrical average) the average of the sinus equals one half...

Of course the same value goes for the cosine because of symmetry
while the tangent should have a geometric average value of 1

$$\lim_{n\to\infty} \sqrt[n]{\tan(\frac{\pi/2}{n}) \cdot \tan(\frac{2\pi/2}{n}) \cdot \tan(\frac{3\pi/2}{n}) \cdots \tan(\frac{(n-1)\pi/2}{n})} = 1$$

Technical remark 1: With sin we took the right hand side of each
partition, with cos we must take the left hand side while with tan
both the zero and the value ∞ need to be avoided.
But if we use the midpoints of our subintervals from the partition
instead of right/left endpoints not only can you write all 3 limits
in the same way but you also get better convergence.
Example for the sin; we go with steps size five degrees and the
result is using the 18 endpoint values

$$\sqrt[18]{\sin 5^\circ \cdot \sin 10^\circ \cdot \sin 15^\circ \cdots \sin 90^\circ} \approx 0.5631$$

But if we use midpoint values and 9 steps of 10 degrees each we get

$$\sqrt[9]{\sin 5^\circ \cdot \sin 15^\circ \cdot \sin 25^\circ \cdots \sin 85^\circ} \approx 0.5196$$

You see that using midpoints nicely speeds up the convergence.

Technical remark 2: For the tan the limit, using midpoints, is also 1 when you do not use the $\sqrt[n]{}$.
This is very easy to see:
For sin and cos the highschool math says

$$\sin\left(\frac{\pi}{2} - x\right) = \cos(x) \text{ and } \cos\left(\frac{\pi}{2} - x\right) = \sin(x)$$

That gives

$$\tan\left(\frac{\pi}{2} - x\right) = \frac{1}{\tan(x)}$$

so if you use midpoints for every partition this is already 1.

I also would like to say a few things about the product integral of the identity function.
So let's first calculate $\prod_1^x t^{dt}$ and $\prod_a^b t^{dt}$.
Of course we do it Reinko style because everybody knows the primitive of the log function.
Here we go

$$\begin{aligned}\textstyle\prod_1^x t^{\,dt} = \exp\left(\int_1^x \log t\ dt\right) \quad &= \exp\left(t\log t - t|_1^x\ \right)\\ &= \exp\left(x\log x - x + 1\ \right)\\ &= x^x \cdot e^{-x+1}\end{aligned}$$

And now for any interval $[a, b]$ in the positive real numbers

$$\begin{aligned}\textstyle\prod_a^b t^{\,dt} = \exp\left(\int_a^b \log t\ dt\right) \quad &= \exp\left(t\log t - t|_a^b\ \right)\\ &= \exp\left(b\log b - b - (a\log a - a)\ \right)\end{aligned}$$

That leads to the conclusion that

$$\prod_a^b t^{\,dt} = \exp\left(b\log b - b - (a\log a - a)\ \right) = \frac{b^b \cdot e^{-b}}{a^a \cdot e^{-a}}$$

It is a well known and rather **fundamental fact** that if $a = 0$ we have $a^a = 1$ and $e^a = 1$. Furthermore, **just as fundamental** we know that $a \overset{\lim}{\downarrow} 0$ from $a\log a = 0$
Therefore if we fill in $a = 0$ and $b = n \in \mathbb{N}$
we would get

$$\prod_0^n t^{\,dt} = \exp\left(x\log x - x\ \right) = n^n \cdot e^{-n}$$

So with $n = 1$ this is

$$\prod_0^1 t^{\,dt} = \exp\left(1 \cdot \log 1 - 1\ \right) = \frac{1}{e}$$

A few pages above we observed that infinite product with the sin function, with the identity function we also have a nice limit on an infinite product.
Here we go:
Given the half-open interval $< 0, 1]$ we split it in n equal parts of size $\triangle_n = 1/n$. The Riemann product that converges to this product integral $\prod_0^1 t^{dt}$ is

$$\lim_{n\to\infty} \prod_{i=1}^{n} \left(\frac{i}{n}\right)^{\triangle_n} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n}\cdot\frac{2}{n}\cdot\frac{3}{n}\cdots\frac{n-1}{n}\cdot 1} = \frac{1}{e}$$

Well a few updates ago in some appendix in the writings to the higher dimensional complex numbers I gave an alternative definition for the number e.
That alternative definition was given as

$$\lim_{n\to\infty} \sqrt[n]{\frac{n^n}{n!}} = e$$

So now you have a nice look in my **math kitchen** because in the appendix I tied this definition to the famous Stirling formulae

$$n! \approx \sqrt{2\pi n} \cdot n^n e^{-n}$$

but in reality I only used an old result from the product integral of the identity function...

Let's recall a bit about the **geometric mean of quantum numbers**. In the update about magnetic monopoles in the writings on higher complex numbers we observed that for spin-half particles they always calculated for $S = \frac{1}{2}, 1\frac{1}{2}, ...$ and whole number quantum numbers like $l = 1, 2, 3, ...$

$$\sqrt{S(S+1)} \text{ and } \sqrt{l(l+1)}$$

Right now the physical importance is not relevant, we observed a Pythagorean relation between these geometrical means, the number 1/2 and a whole number on the hypotenuse. For example is $S = 5\frac{1}{2}$ than $S(S+1) = 35\frac{3}{4}$ so

$$35\frac{3}{4} + \left(\frac{1}{2}\right)^2 = 36 \ (= 6^2)$$

And there is a dual relation, if for example $l = 5$ then $l(l+1) = 30$ and

$$\sqrt{30 + \frac{1}{4}} = 5.5$$

Simple conclusion:
This way quantum half numbers give quantum whole numbers and quantum whole numbers give quantum half numbers.
To be honest with you: at present date I do not have a clue if this is just some **mathematical curiosity** or that this has some deeper physical meaning.

Now what has stuff like this to do with product integration? For normal or additive integration we know that if we take the integral over an interval of length 1, we get the average value of the function on that interval

$$\int_x^{x+1} f(t)\, dt = \text{ average value of } f(x) \text{ over the interval } [x, x+1]$$

So let us check if the product integral of the identity function has similar properties. It is not hard to check that

$$\prod_n^{n+1} t^{\,dt} = \frac{(n+1)^{n+1}}{n^n \cdot e}$$

With $n = 5$, just an example, this gives

$$\frac{6^6}{5^5 \cdot e} \approx 5.4924 \text{ while } \sqrt{5 \cdot 6} = \sqrt{30} \approx 5.4772$$

So it is not exactly equal but it surely is asymptotic the same because if we raise the value of n a little bit to $n = 15$ we have

$$\frac{16^{16}}{15^{15} \cdot e} \approx 15.4973 \text{ while } \sqrt{15 \cdot 16} = \sqrt{240} \approx 15.4919$$

From standard mathematical theory we know that the geometric mean of two numbers converges to the additive mean. The **relevant stuff** is that the product integral converges much faster.
But, it has to be remarked that I am far to lazy to produce solid proof to that statement because as you could read in the introduction on page 1:
The **precious professional professors** are only interested into linearisation of things because a strait line is something they understand. So even if product integration is a far more efficient mathematical tool, they will not use it because they do not understand it.
And after all, we live in a free world, and in a free world the best outcome is guaranteed if people use things they understand...

We should never forget that.

But all in all, the closing limit of these pages about product integration is hopefully a bit helpful in promoting the **product integral** as a mathematical subject in it's own right

$$\lim_{n\to\infty} \frac{(n+1)^{n+1}}{n^n \cdot e} - n = \frac{1}{2}$$

Appendix 1: Raabe's formulae for the log of the gamma function.

Three months after I finished the above work on the 10 styles of product integration I came across this Raabe formulae for the gamma function. I have absolutely no idea how Raabe found this result but if you see it you understand it screams for a nice product integral version. Here is what Mr. Raabe (an Austrian math guy) had found:

$$\int_a^{a+1} \log \Gamma(x) dx = \frac{1}{2} \log 2\pi + a \log a - a$$

There are plenty of proofs out there on the internet, so I will not repeat those proofs because that is already known.

Of course the gamma function is given by it's usual definition:

$$\Gamma(t) = \int_0^\infty x^{t-1} e^{-x} dx.$$

So at a first glimpse taking the product integral of $\Gamma(t)$ looks hard because what is the log of this???
Well now we simply apply Raabe's formulae;

If we take the product integral of the gamma function we get

$$\prod_a^{a+1} \Gamma(x)^{dx} = \exp\left(\int_a^{a+1} \log \Gamma(x) dx \right) \stackrel{1}{=} \sqrt{2\pi} a^a \cdot e^{-a}.$$

Where at $\stackrel{1}{=}$ we apply the result of Raabe.
Of course we are not finished here because

the product integral of the identity function $f(x) = x$ is given by

$$\prod_0^a x^{dx} = a^a e^{-a}$$

hence the product integral of the gamma function can be written as

$$\prod_a^{a+1} \Gamma(x)^{dx} = \sqrt{2\pi} \cdot \prod_0^a x^{dx}.$$

Remark that since the length of the integration interval is 1, this product integral returns the average value in the sense of multiplication. That is also known as the geometric average.

This is a beautiful result; look at the integration ranges: The gamma functions goes from a to $a+1$ and all you have to do is multiply $\sqrt{2\pi}$ with the product integral of the identity function over the interval $[0, a]$.
This result is important because the identity function is one of the most easy to product-integrate functions there are.
Furthermore the gamma function is a smooth function that goes through all factorials $n!$ and this Raabe thing nicely connects the stuff...

Hopefully we are now really at the end of this file.