An overview of exponential circles and exponential curves in $\mathbb{C}, \mathbb{R}^3, \mathbb{C}^3$ and $\mathbb{R}^5$.

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An overview of exponential circles and exponential curves in $\mathbb{C}, \mathbb{R}^3, \mathbb{C}^3$ and $\mathbb{R}^5$.

I heard someone say that in the year 1748 Leonhard Euler discovered or published the very first exponential circle any human had found.
It is known as Euler's formulae or the Euler identity and even in popular culture it is famous.
There are even movies where you hear people saying weird stuff like:
The number e to the power iπ equals minus one so God exists.
Needless to say this is completely bogus logic, it is more like one of those religious statements that are so often riddled with bogus logic.
But let's not crack down on religious people, very often scientists do that but I think that religion is an important function of the human brain. Don't forget that religion also channels all kinds of moral behaviour, moral behaviour is also a function of the human brain, and science does not channel moral stuff in any way.

Ok let's sidestep all religious stuff and look at the Euler identity once more

$$e^{it} = \cos t + i \sin t \text{ for } t \in \mathbb{R}$$

This is great stuff because it relates exponentials
to trigonometry and all those sum and difference
formula's for trigonometry is simplified significantly.
For example the Euler identity also says

$$e^{it} \cdot e^{is} = \cos(t+s) + i\sin(t+s)$$

And from that it is always easy for the human brain
to reconstruct all those sum & difference formula's
from the field of trigonometry.

So far for the complex plane $\mathbb{C}$.
In $\mathbb{R}^3$ we found two exponential circles
depending on what kind of multiplication we used.
Both exponential circles were written as

$$f(t) = e^{\tau t}.$$

And although that looks simple in notation, the
math is found in the $'\tau'$ stuff.
In 3 dimensions with the complex multiplication,
both imaginary components had to be the same.
Therefore τ was like

$$\tau = (j + j^2)\frac{2\pi}{3\sqrt{3}}$$

We observe that τ looks a bit complicated
but therefore $f(t)$ was so simple to understand
because it simply rotated over the dimensions.

$$f(0) = 1, f(1) = j \text{ and } f(2) = j^2$$

And we identified the set $\{1, j, j^2\}$ with the
basis vectors (x, y, z) in $\mathbb{R}^3$ about 23 years
ago as follows:

$$\begin{aligned} 1 &= (1,0,0) \\ j &= (0,1,0) \\ j^2 &= (0,0,1) \end{aligned}$$

In the update from 30 Jan this year I describe an
important principle known as the **pullback principle**.
With the pullback principle you can calculate the

periods of exponential circles and curves in
higher dimensions while using the complex plane $\mathbb{C}$.

Let me illustrate that for the τ standing above:
In $\mathbb{R}^3$ we have for the complex multiplication
that $j^3 = -1$ but in the complex plane $\mathbb{C}$
these are roots of minus unity.
We now 'pull back' our $j + j^2$ to $\mathbb{C}$ via
the famous map φ and map both imaginary parts
as follows

$$\varphi(j) = e^{i\pi/3} \text{ and } \varphi(j^2) = e^{2i\pi/3}$$

And then we simply add

$$\varphi(j) + \varphi(j^2) = e^{i\pi/3} + e^{2i\pi/3} = \sqrt{3} \cdot i$$

We do this because we want to know the period of
$e^{(j+j^2)t}$ in $\mathbb{R}^3$ and if we pull that back to $\mathbb{C}$ we get $e^{\sqrt{3} \cdot it}$
and the latter has of course a period in time of

$$\frac{2\pi}{\sqrt{3}}$$

That explains why I choose τ to be

$$\tau = (j + j^2)\frac{2\pi}{3\sqrt{3}}$$

so that it was guaranteed that the exponential circle
$f(t) = e^{\tau t}$ has a period of 3.

Let us do **exactly the same analysis** for the
circular multiplication in $\mathbb{R}^3$, so now
the rotation over the basis vectors goes like $j^3 = 1$.
Now the triplet $\{1, j, j^2\}$ are the roots of unity,
so if we pull the imaginary parts back to $\mathbb{C}$ we get

$$\varphi(j) = e^{2i\pi/3} \text{ and } \varphi(j^2) = e^{4i\pi/3}$$

In order to get something completely imaginary in $\mathbb{C}$
we now have to subtract stuff like in

$$\varphi(j) - \varphi(j^2) = e^{2i\pi/3} - e^{4i\pi/3} = \sqrt{3} \cdot i$$

This **perfectly explains** why for the circular multiplication a good version of τ looks like

$$\tau = (j - j^2)\frac{2\pi}{3\sqrt{3}}$$

I want to close the overview for $\mathbb{R}^3$ so as a final end to this subpart let's write out the explicit coordinate functions in a way just like Leonhard Euler did in the year 1748:

$$e^{it} = \cos t + i \sin t$$

You can view this is a moving point on the unit circle in $\mathbb{C}$ that is projected on the two coordinate axis. In the case of $\mathbb{R}^3$ the exponential circles $e^{\tau t}$ make the same angle with all three coordinate axis.
Since we have made *tau* a little bit more difficult, the period is now 3 time units and if we want to capture that with cosine functions, the exponential circle for the circular multiplication becomes

$$f(t) = \frac{1}{3} + \frac{2}{3}\cos\left(\frac{2\pi}{3}t\right) + \frac{j}{3} + \frac{2j}{3}\cos\left(\frac{2\pi}{3}(t-1)\right) + \frac{j^2}{3} + \frac{2j^2}{3}\cos\left(\frac{2\pi}{3}(t-2)\right).$$

Recall that for the circular multiplication we go like $1 \to j \to j^2 \to 1$ etc etc.

For the complex version I made it go like
$1 \to j^2 \to -j \to 1$ etc etc.
That means in terms of basis vectors it goes like
$(1, 0, 0) \to (0, 0, 1) \to (0, -1, 0) \to 1$ etc etc.
So the $y-$coordinate behave differently because the determinant of the matrix representation must stay one all the time.
And in $\mathbb{R}^3$ we have $\det(-X) = -\det(X)$ because the dimension is an odd number.
After having said that, stuff should now look like

$$f(t) = \frac{1}{3} + \frac{2}{3}\cos\left(\frac{2\pi}{3}t\right) - \frac{j}{3} - \frac{2j}{3}\cos\left(\frac{2\pi}{3}(t-2)\right) + \frac{j^2}{3} + \frac{2j^2}{3}\cos\left(\frac{2\pi}{3}(t-1)\right).$$

For myself speaking, I prefer to write it a little bit more compact also stressing that in $\mathbb{R}^3$ the number α is always the center of the exponential circles. Doing so gives

$$f(t) = \alpha + \frac{2}{3}\cos\left(\frac{2\pi}{3}t\right) + j\frac{2}{3}\cos\left(\frac{2\pi}{3}(t-1)\right) + j^2\frac{2}{3}\cos\left(\frac{2\pi}{3}(t-2)\right).$$

For the complex multiplication in 3D this would give

$$f(t) = \alpha + \frac{2}{3}\cos\left(\frac{2\pi}{3}t\right) - \frac{2j}{3}\cos\left(\frac{2\pi}{3}(t-2)\right) + \frac{2j^2}{3}\cos\left(\frac{2\pi}{3}(t-1)\right).$$

The very important **sphere-cone** equation dictates what properties these coordinate functions have and because the exponential circles are on the unit sphere in $\mathbb{R}^3$ the sum of squares add up to one.
If for simplicity we write $f(t) = c_0(t) + jc_1(t) + j^2c_2(t)$ we have with even more simplicity suppressing the dependence on t

$$c_0^2 + c_1^2 + c_2^2 = 1$$

The coordinate functions must also obey the equations of the cones. And there is an additional equation to be full filled: the endpoints of the three basis vectors span up a plane.
For the circular & complex multiplication that would give additional equations like

$$\begin{cases} x + y + z = 1 \text{ (circular multiplication)} \\ x - y + z = 1 \text{ (complex multiplication)} \end{cases}$$

For example, if we restrict ourselves to the circular version we get

$$\begin{cases} c_0^2 + c_1^2 + c_2^2 = 1 \\ c_0 + c_1 + c_2 = 1 \end{cases}$$

And by all standards, that is a cute result...

So far for $\mathbb{R}^3$, let's now go to $\mathbb{C}^3$.

There are always two ways to look at this,
you can say

$$\mathbb{C}^3 = \mathbb{R}^3 + i\mathbb{R}^3 \text{ so that}$$
$$Z = X + iY \text{ with } X, Y \in \mathbb{R}^3$$

And the other way is

$$\mathbb{C}^3 = \mathbb{C} + j\mathbb{C} + j^2\mathbb{C} \text{ so that}$$
$$Z = z_0 + jz_1 + j^2 z_2 \text{ with } z_0, z_1 \text{ and } z_2 \in \mathbb{C}$$

Depending on how you view it you get slightly different but equivalent matrix representations for $Z \in \mathbb{C}^3$.
As usual we look at the two possible ways for multiplying j;
$j^3 = 1$ is again the circular multiplication and
$j^3 = -1$ is the complex version.

During my petite investigations into the higher dimensional complex numbers I often used the applets from calculator-fx.com. But for an unknown number of months this website is off-line.

The applets from that website were highly reliable, especially the applet for taking the log of a matrix was good because at other websites you often got strange results that clearly could not be right...
So I cannot give you a screenshot for further validation of our exponential curves on $\mathbb{C}^3$ but if you have access to a good applet or that expensive software that has good routines in it, you can take the log of the next matrix representation for j.
Let's take the complex multiplication so $j^3 = -1$ and we replace in the standard 3D matrix all entries with small 2×2 matrices that resemble the matrix representation on the complex plane.

So we get a $6^\times 6$ matrix

$$M(j) = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Again important 1: View this as an 3×3 matrix with complex entries, but most applets do not allow for complex numbers from the complex plane $\mathbb{C}$.
By the way, this matrix is also the matrix representation for j from $\mathbb{R}^6$ as you see on inspection.
Again important 2: You should use the complex multiplication because if you use the circular multiplication the applet views the 6×6 matrix as a 3×3 made of 2×2 things. Why this is I don't know.

Anyway, to make a long story short; if you have such an applet for matrix log, the result is that when viewed inside $\mathbb{C}^3$ we have

$$\log j = \tau/2 + i\pi\alpha$$

This goes **only for the complex multiplication** on $\mathbb{R}^\mathbf{3}$ because of determinant problems, the circular version simply has

$$\log j = \tau \text{ because there } \det M(j) = 1$$

Of course both multiplications have their own version of τ and α from $\mathbb{R}^3$.
So this is our new $'\tau'$ suited for $\mathbb{C}^3$, but it would be highly confusing to write stuff like

$$\tau = \tau/2 + i\pi\alpha$$

where on the left hand side it is τ from $\mathbb{C}^3$ and on the right hand side the τ from $\mathbb{R}^3$.
Also I always try to avoid all those difficult to read indices like $\tau_{\mathbb{C}^3}$ or $\tau_{\mathbb{R}^3}$ that **professional professors**

use so often...
At this point in time it looks reasonable that every time we go from some $\mathbb{R}^n$ via complexification to $\mathbb{C}^n$, to use the symbol θ because that also starts with a t.
By the way, **do you know** why I named it $\tau-$**calculus** in the first place? That is because in my Dutch home language τ rhymes on touw and touw means rope.
The 'rope' is used to come from 1 to j when calculating the $\log j$ like in

$$\tau = \int_1^j \frac{1}{X} \, dX$$

Ok, now we have $\theta-$**calculus** and I claim that as we view $\mathbb{C}^3$ as a complexification of the two multiplications on $\mathbb{R}^3$, that we have

$$\theta = \tau/2 + i\pi\alpha$$

Therefore inside $\mathbb{C}^3$ there are two exponential curves that can be written as

$$f(t) = e^{\theta t} = e^{(\tau/2+i\pi\alpha)t}$$

In the update on a new Cauchy integral formulae using the exponential curve on $\mathbb{C}^3$ from 18 Jan this year, see page five, I made a factorization of the exponential curve. But that was for the complex multiplication.
In this overview I will give them for both the complex & circular multiplication.
It is not hard to see that

$$f(t) = e^{\theta t} = e^{(\tau/2+i\pi\alpha)t} = e^{\tau t/2} \cdot e^{i\pi\alpha t}$$

And it is also not hard to see that the 'old' exponential circle $e^{\tau t}$ is now run at half the original speed by dividing τ by 2.
First we look at the circular multiplication.
This means that
$e^{\tau/2}$ is halfway 1 and j,

$e^{3\tau/2}$ is halfway j and j^2,
$e^{5\tau/2}$ is halfway j^2 and 1.

Let us recall what we know about the opposite points on the exponential circle in $\mathbb{R}^3$:
First the general geometric stuff; let A and B be two points or two vectors in an arbitrary vector space, what point is halfway A and B?

This is high school math, if we denote the center with C we get

$$C = \frac{A+B}{2}$$

This means that given an A and a C, we can find B;

$$B = 2C - A \text{ or vice versa } A = 2C - B$$

That gives:
$e^{\tau/2}$ is halfway 1 and j and is opposite to j^2,
$e^{3\tau/2}$ is halfway j and j^2 and is opposite to 1,
$e^{5\tau/2}$ is halfway j^2 and 1 and is opposite to j.

Since the number α is the center of both exponential circles in $\mathbb{R}^3$ we have:

$e^{\tau/2} = 2\alpha - j^2$
$e^{3\tau/2} = 2\alpha - 1$
$e^{5\tau/2} = 2\alpha - j$.

We now have the left half of the factorization of $f(t) = e^{\tau t/2} \cdot e^{i\pi\alpha t}$.

The right hand side is very surprising, I still have to find a good analytical proof for it but with any good applet for the exponential matrix you can find it. But that is not a real proof, anyway call me old fashioned but if I can avoid machines in a proof I always do that...

But there is a screen shot on my computer
that says using some applet for exponential
matrices:

$$e^{i\pi\alpha} = -(2\alpha - 1)$$

As a direct consequence of that we have
this highly familiar looking

$$e^{2i\pi\alpha} = 1$$

Just like in the complex plane $\mathbb{C}$ we have
that $e^{2i\pi} = 1$, in $\mathbb{C}^3$ we need an extra factor α.
Remarkable this goes for both multiplications.

Let's **craft a proof** for this cute looking
identity: $e^{2i\pi\alpha}$ is the square of $e^{i\pi\alpha}$,
so all we have to do is to square $-(2\alpha - 1)$.
Using the fundamental property $\alpha^2 = \alpha$ the
simple calculation becomes
$[-(2\alpha - 1)]^2 =$
$(2\alpha - 1)^2 =$
$4\alpha^2 - 4\alpha + 1 = 1$. Qed.

Ok, it is about time to write down the table
with six values for the exponential curve
$f(t) = e^{\tau t/2} \cdot e^{i\pi\alpha t}$.
Here we go, for the circular multiplication we
use the property $j\alpha = \alpha$;

$$\begin{aligned}
&f(0) = e^0 \cdot e^0 = 1 \\
&f(1) = e^{\tau/2} \cdot e^{i\pi\alpha} = (2\alpha - j^2) \cdot -(2\alpha - 1) = -j^2 \\
&f(2) = e^{\tau} \cdot e^{2i\pi\alpha} = j \cdot 1 = j \\
&f(3) = e^{3\tau/2} \cdot e^{3i\pi\alpha} = (2\alpha - 1) \cdot -(2\alpha - 1) = -1 \\
&f(4) = e^{2}\tau \cdot e^{4i\pi\alpha} = j^2 \cdot 1 = j^2 \\
&f(5) = e^{5\tau/2} \cdot e^{5i\pi\alpha} = (2\alpha - j) \cdot -(2\alpha - 1) = -j \\
&f(6) = e^{3\tau} \cdot e^{6i\pi\alpha} = j^3 \cdot 1 = 1
\end{aligned}$$

So the period is 6, $\forall t \in \mathbb{R}$ $f(t + 6) = f(t)$.

Another way of calculating these values is via
remarking that since $f(t)$ is an exponential

curve it has the property that for $k \in \mathbb{N}$

$$f(k) = e^{\theta k} = f(1)^k \text{ so that } f(k) = (-j^2)^k$$

and using $j^3 = 1$ we can break down the higher powers of j. Let's do it

$$\begin{aligned}
f(0) &= (-j^2)^0 = 1 \\
f(1) &= (-j^2)^1 = -j^2 \\
f(2) &= (-j^2)^2 = j^4 = j \\
f(3) &= (-j^2)^3 = -j^6 = -1 \\
f(4) &= (-j^2)^4 = j^8 = j^2 \\
f(5) &= (-j^2)^5 = -j^{10} = -j \\
f(6) &= (-j^2)^6 = j^{12} = 1
\end{aligned}$$

You see: both methods give the same results.

Now we do the same for the **complex multiplication** on $\mathbb{C}^3$. In the update of 18 Jan 2014 about a new Cauchy integral I already made a table with the factors of the exponential curve listed.
Let me simply cut & paste it here:

t	$f(t)$	$e^{t\tau/2}$	$e^{ti\pi\alpha}$
0	1	1	1
1	j	$2\alpha + j$	$-(2\alpha - 1)$
2	j^2	j^2	1
3	-1	$2\alpha - 1$	$-(2\alpha - 1)$
4	$-j$	$-j$	1
5	$-j^2$	$2\alpha - j^2$	$-(2\alpha - 1)$

We observe the beautiful behaviour of the factor $e^{ti\pi\alpha}$ in the last column:
It alternates between 1 (that has determinant plus 1) and $-(2\alpha - 1)$ (that has determinant minus 1).

As an example of how stuff works let's check $f(5)$:

$$f(5) = e^{5\tau/2} \cdot e^{5i\pi\alpha} =$$

$$(2\alpha - j^2) \cdot -(2\alpha - 1) = -(4\alpha^2 - 2\alpha - j^2 2\alpha + j^2) =$$

$$-(4\alpha - 2\alpha - 2\alpha + j^2) = -j^2$$

Let's also check $f(1)$;

$$f(1) = e^{\tau/2} \cdot e^{i\pi\alpha} =$$

$$(2\alpha + j) \cdot -(2\alpha - 1) = -(4\alpha^2 - 2\alpha + 2j\alpha - j) \stackrel{1}{=}$$

$$-(4\alpha - 2\alpha - 2\alpha - j) = j$$

Of course with the complex multiplication the number α behaves slightly different, now it's basic properties used in $\stackrel{1}{=}$ are

$$\alpha^2 = \alpha$$
$$j\alpha = -\alpha$$

It now makes no sense at all to make a second list and see if the outcome is the same because already $f(1) = j$ contrary to the circular multiplication where $f(1) = -j^2$.
This small difference is caused by the fact that for the complex multiplication in $\mathbb{R}^3$ the successive powers of j automatically go through the pluses and minuses of all basis vectors because of $j^3 = -1$.

Before we go to the two five-dimensional exponential curves I would like to write down the sphere-cone equation for $\mathbb{C}^3$.
I do that because I do not have a good proof that the second factor $e^{i\pi\alpha t}$ is an exponential circle.
Yet the way the sphere-cone equation looks

suggests strongly that indeed these are
exponential circles.
The oldest sphere-cone equation is the
equation for the unit circle in $\mathbb{C}$
namely

$$z\overline{z} = 1 \text{ where } z \in \mathbb{C}$$

and everybody knows that
$(x + iy)(x - iy) = x^2 + y^2 = 1$ defines
the Euler circle e^{it}.

Ok, let's write it down for both multiplications
at the same time, of course when taking the
conjugate stuff is different. So kepp that
in mind.
For $Z \in \mathbb{C}^3$ write

$$\begin{array}{l} Z = X + iY \text{ with } X, Y \in \mathbb{R}^3 \text{ so that} \\ \overline{Z} = \overline{X} - i\overline{Y} \end{array}$$

[Again: $\overline{X}$ depends on the actual style
of multiplication, complex or circular.]
The sphere-cone equation now becomes

$$\begin{array}{l} Z\overline{Z} = 1 \Longleftrightarrow \\ (X + iY)(\overline{X} - i\overline{Y}) = \\ X\overline{X} + Y\overline{Y} + i(Y\overline{X} - X\overline{Y}) = 1 \end{array}$$

That would give a system of two equations

$$\left\{ \begin{array}{l} X\overline{X} + Y\overline{Y} = 1 \\ \overline{X}Y - X\overline{Y} = 0 \end{array} \right.$$

The second equation says that $\overline{X}Y = X\overline{Y}$ and
that is only possible if $\overline{X}Y = X\overline{Y}$ so we **want
to know** when this is true in $\mathbb{R}^3$.

On the complex plane $\mathbb{C}$ a number z is also it's own
conjugate $\overline{z}$ iff $z \in \mathbb{R}$ because the imaginary
component must be zero on $\mathbb{C}$.
In $\mathbb{R}^3$ the situation is completely different;

Case 1: Circular conjugate.
For $X = x + yj + zj^2$ the conjugate is given by
$\overline{X} = x + zj + yj^2$. Hence $X = \overline{X}$ iff $y = z$,
so when both imaginary components are equal.
If we use the very important **pullback principle**
that is the map $\varphi : \mathbb{R}^3 \longrightarrow \mathbb{C}$ we observe

$$\varphi(j + j^2) = \varphi(j) + \varphi(j^2) = e^{2i\pi/3} + e^{4i\pi/3} = -1 \in \mathbb{R}$$

And since the beautiful science known as math has all
that **internal coherence** in it, the conjugates on $\mathbb{R}^3$
nicely relate to the conjugates on $\mathbb{C}$ via the pullback φ.

Case 2: Complex conjugate.
For $X = x + yj + zj^2$ the conjugate is given by
$\overline{X} = x - zj - yj^2$. Hence $X = \overline{X}$ iff $y = -z$,
so when both imaginary components have opposite sign.
For the 3D complex multiplication we have $j^3 = -1$ and if
we pull that back to $\mathbb{C}$ we have $\varphi(j) = e^{i\pi/3}$ and $\varphi(j^2) = e^{2i\pi/3}$.
And the only linear combinations of j and j^2 that make this a
real number are $j - j^2$ or $-j + j^2$.

So now we know when $\overline{X}Y - X\overline{Y} = 0$ for $X, Y \in \mathbb{R}^3$.

This gives hope that indeed the second
factor with $i\pi\alpha$ in the exponent is indeed an
exponential circle but for the time being I would
like some more additional clues & a perfect
proof.

One day later: problem solved; the stuff
known as $f(t) = e^{i\pi\alpha t}$ is indeed a circle.
All you need is to realize that $\alpha^2 = \alpha$, well
we have seen the projective nature of α before.
After my humble opinion the projective behaviour
of α (and also τ) make the higher dimensional
complex numbers perfect candidates for quantum
mechanics.
Anyway, let P be any projection such that

$P^2 = P$, in that case

$$e^{tP} = 1 + P\left(e^t - 1\right)$$

If you have never found that, it is extremely easy to prove using the standard definition for the exponential function

$$e^{tP} = \sum_{n=0}^{\infty} \frac{t^n P^n}{n!} = 1 + P \sum_{n=1}^{\infty} \frac{t^n}{n!} = \text{ etc etc.}$$

Now we will do exactly that for $e^{i\pi\alpha t}$, here we go:

$$e^{i\pi\alpha t} = \sum_{n=0}^{\infty} \frac{(i\pi\alpha t)^n}{n!} =$$

$$1 + \alpha \sum_{n=1}^{\infty} \frac{(i\pi t)^n}{n!} =$$

$$1 + \alpha(e^{i\pi t} - 1)$$

Of course only **professional professors** in math will deny this is a circle, but hey: they live their lives & I live mine...
But let us not get **emotionally disrupted** by a bunch of **overpaid incompetents**. Because we are now going to calculate the middle point (the center) of this exponential circle.
That is very easy since $\mathbb{C}^3$ is also a vector space and if you know the coordinates of two opposite points of a circle, all you have to do is take the average of these two points.
Since the period is 2, $f(t+2) = f(t)$ we can take the average of $f(0) = 1$ and $f(1) = 1 - 2\alpha$. That gives

$$\frac{1 + (1 - 2\alpha)}{2} = 1 - \alpha$$

Why the hell would such a center be important to calculate? Well we now have solved nicely and without using machines that $e^{i\pi\alpha t}$ is an exponential circle.
But that creates a new unsolved problem:
Why do all exponential circles and curves

always have a non-invertible number as their center?

Let's sum up what we found until now:
Two exponential circles in $\mathbb{R}^3$, written as $f(t) = e^{\tau t}$ where $\tau = \log j$ for the circular multiplication with $j^3 = 1$ and $\tau = \log j^2$ for the complex multiplication with $j^3 = -1$.

Going from $\mathbb{R}^3$ to $\mathbb{C}^3$ we found two more exponential circles, but now related to the numbers α, written as $g(t) = e^{2i\pi\alpha t}$. An alternative representation using the property $\alpha^2 = \alpha$ gives us

$$g(t) = 1 + \alpha \left(e^{2i\pi t} - 1 \right)$$

If we divide both exponents by 2 and we multiply the f and g circles we get two new exponential curves (they cannot be circles) and these can be written as

$$h(t) = e^{\theta t} = e^{(\tau/2 + i\pi\alpha)t}$$

and this curve $h(t)$ has the property that it passes through all plusses and minussus of the basis vectors; so through $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$ and $(0, 0, \pm 1)$.

Exponential circles and curves in $\mathbb{R}^5$ and $\mathbb{C}^5$. Both numbers α belonging to the circular and complex multiplication on $\mathbb{C}^5$ give an exponential circle namely

$$f(t) = e^{i\pi\alpha t}.$$

Here $\mathbb{C}^5 = \mathbb{R}^5 + i\mathbb{R}^5$, these $f(t)$ have most of the time an 'imaginary component' related to the i that turns $\mathbb{R}^5$ into a 10-dimensional structure $\mathbb{C}^5$.

The α numbers are just like in $\mathbb{R}^3$ given by
the squares of all the basis vectors $\{1, j, j^2, j^3, j^4\}$
where we make the usual identification like $j = (0, 1, 0, 0, 0)$.
Let me skip the calculation, you can add
the squares for yourself if you like.
The circular multiplication has

$$\alpha = \frac{1 + j + j^2 + j^3 + j^4}{5}$$

and the complex version has

$$\alpha = \frac{1 - j + j^2 - j^3 + j^4}{5}.$$

It would be an **important but easy exercise**
to write out all the details in order to prove that
also in $\mathbb{R}^5$ we have

$$\alpha^2 = \alpha$$

so that just like in $\mathbb{R}^3$ we have, with the
help of Leonhard Euler, the result that

$$e^{i\pi\alpha t} = 1 + \alpha \left(e^{i\pi t} - 1\right)$$

Now we turn to $\mathbb{R}^5$, how did I solve that
relatively hard problem? Because all I had was that
numerical output from an internet applet for the log
of a matrix.
Well, with my hand-calculator I just looked at the
quotient of the numbers and that looked familiar:
That quotient was the negative root of the golden
ratio. So we repeat from the update from 20 Jan of
this year the golden ratio:
**As the small stands to the larger part, so
stands the larger part to the whole.** Or, as
fractions with s = small and g = the greater part

$$\frac{g}{s} = \frac{g + s}{g} \text{ so that } g^2 = sg + s^2$$

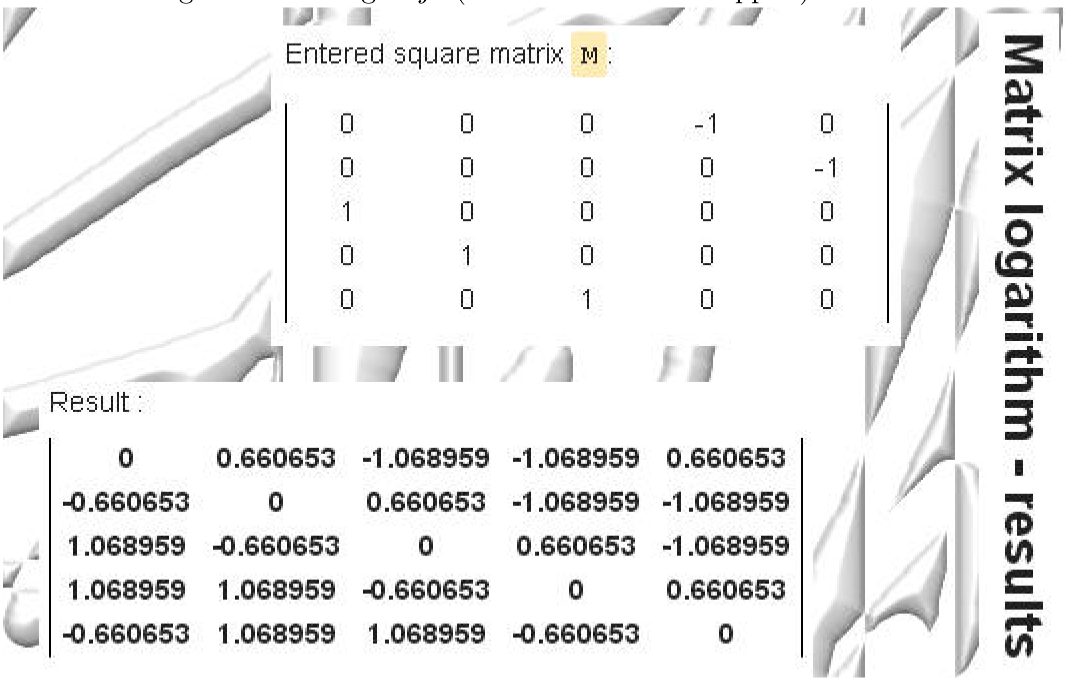
Solving this gives the two golden ratio's

$$g = \frac{1 \pm \sqrt{5}}{2} s$$

The factor that relates the small and larger part of the golden ratio is now found back in $\log j^2$ and $\log j$.
Also in $\mathbb{R}^5$ the determinant has to be one so therefore we need to take the log of j^2 for the complex multiplication (with $j^5 = -1$).
Below you see a picture with a screen shot with the results for $\log j^2$.

As you see if you evaluate the quotient

Figure 1: The log of j^2 (done with a matrix applet)



$$\frac{-0.660653}{1.068959} \approx -0.618034 \text{ where also } \frac{1-\sqrt{5}}{2} \approx -0.618034$$

Of course it is **important to remark** that while I take the negative golden ratio, you can also use the positive golden ratio because they are easily related via

$$\frac{1-\sqrt{5}}{2} \cdot \frac{1+\sqrt{5}}{2} = -1$$

So what will the number τ be? Well in $\mathbb{R}^3$ I did put in all the difficult stuff into the 3D τ in order to get a sleek $e^{\tau t}$ that has a period in time of 3 and a spatial period of 3τ meaning

$$\forall t \in \mathbb{R}, \ \ e^{\tau(t+3)} = e^{\tau t} \text{ and } \forall X \in \mathbb{R}^3, \ \ e^{X+3\tau} = e^X$$

But in $\mathbb{R}^5$ it is better to keep τ as simple & sleek as possible so that we later can scale the 'speed of time' in order to get a period in time of 5.
So the next version of a number τ does not have $e^\tau = j^2$ for the complex multiplication, or $e^\tau = j$ for the circular multiplication.
But later we will correct the 'speed of time'.
The relevant detail is of course that the ratio's between the four imaginary components is respected!

The complex version with $j^5 = -1$ gives a τ of

$$\tau = \frac{1-\sqrt{5}}{2} j + j^2 + j^3 + \frac{1-\sqrt{5}}{2} j^4$$

The circular version with $j^5 = 1$ gives a τ of

$$\tau = j + \frac{1-\sqrt{5}}{2} j^2 - \frac{1-\sqrt{5}}{2} j^3 - j^4$$

It is important to understand why both versions of τ look like the look. If we use the pull back map $\varphi : \mathbb{R}^5 \longrightarrow \mathbb{C}$ we observe that for both τ we have that $\varphi(\tau) =$ purely imaginary on $\mathbb{C}$.
Suppose that $\varphi(\tau) = ai$, in that case the period of the exponential curve equals $2\pi/a$, this is directly caused by the fact that all exponential circles and curves run their curve with a constant speed. Just like the Euler exponential circle $f(t) = e^{i\pi t}$ is done with a constant speed of π length units/ second.

Now we proceed with calculating the pull back of τ,
first for the **complex** multiplication:

$$\varphi(j) = e^{i\pi/5}, \varphi(j^2) = e^{2i\pi/5}, \varphi(j^3) = e^{3i\pi/5} \text{ and } \varphi(j^4) = e^{4i\pi/5}$$

We observe that the cosine parts of $j + j^4$ and of $j^2 + j^3$ cancel out so that $\varphi(\tau)$ will not have a real part.
Therefore, written in degrees, i.e. $180°/5 = 36°$

$$\varphi(\tau) = \left(\frac{1-\sqrt{5}}{2} \sin 36° + \sin 72° + \sin 108° + \frac{1-\sqrt{5}}{2} \sin 144° \right) i$$

This simplifies further because of symmetry in the $y-$axis, namely $\sin 36° = \sin 144°$ and $\sin 72° = \sin 108°$

Beside the golden ratio there is more ancient knowledge to be written down:

$$\sin(36°) = \frac{1}{4}\sqrt{10 - 2\sqrt{5}} \text{ and } \sin(72°) = \frac{1}{4}\sqrt{10 + 2\sqrt{5}}$$

And if we plug that all in and divide by i we get

$$\frac{\varphi(\tau)}{i} = \frac{1-\sqrt{5}}{2} \cdot \frac{1}{2}\sqrt{10 - 2\sqrt{5}} + \frac{1}{2}\sqrt{10 + 2\sqrt{5}}$$

So now we have the a in $\varphi(\tau) = ai$.
The period in time T for $e^{\tau t}$ now becomes $T = 2\pi/a$ or

$$T = \frac{4\pi}{\frac{1-\sqrt{5}}{2} \cdot \sqrt{10 - 2\sqrt{5}} + \sqrt{10 + 2\sqrt{5}}}$$

Hence

$$e^{\tau(t+T)} = e^{\tau t} \text{ where } e^{\tau T} = 1 \text{ and } T \approx 5.344796661$$

Of course, if you want, you can use $e^{\tau t}$ as your exponential curve with a period in time of size T.
But a last small modification will give the result that I prefer namely

$$f(t) = e^{\tau t T/5} \text{ because now}$$

$$f(t+5) = e^{\tau(t+5)T/5} = e^{\tau t T/5 + \tau T} = f(t)$$

This is a thundering result because this goes through all the basis vectors via multiplication with j^2, recall our $\tau = \log j^2$ because the determinant must be one. Just look:

$$\begin{aligned} f(0) &= 1 = (1,0,0,0,0) \\ f(1) &= j^2 = (0,0,1,0,0) \\ f(2) &= j^4 = (0,0,0,0,1) \\ f(3) &= -j = (0,-1,0,0,0) \\ f(4) &= -j^3 = (0,0,0,-1,0) \\ f(5) &= 1 \quad \text{etc etc} \end{aligned}$$

I would like to skip the same calculus for the **circular multiplication** (with $j^5 = 1$) because it is so similar. Beside that, in case you need exponential curves with the circular multiplication it is always better to do it yourself because only that burns it better into your brain...

At the end of this update I want to look at the **sphere-cone equation** on $\mathbb{R}^5$ and show you that the two exponential curves are a solution to that equation. Recall on $\mathbb{C}$ the name is not sphere-cone equation but something like circle equation.
Any it is this one:

$$z\overline{z} = 1 \text{ has solution } x^2 + y^2 = 1 \text{ or } e^{it}$$

On $\mathbb{R}^5$ it is

$$X\overline{X} = 1$$

But if you write that all out, you will see that what you are left with is basically a two-dimensional thing. If you add another equation adding the restriction it has to be in the 4D hyperplane that goes through all the relevant basis vectors, finally you are left with a one-dimensional curve.
So we add the extra restriction

$$x_0 + x_1 + x_2 + x_3 + x_4 = 1 \text{ circular multiplication}$$

$$x_0 - x_1 + x_2 - x_3 + x_4 = 1 \text{ complex multiplication}$$

The reason that you need to add more restrictions is very simple: If you write out all the details in $X\overline{X}$ you will find only 3 different equations. And that is not enough to guarantee a one dimensional solution...
Let me, once more, write down the conjugates for all powers of j for the circular & complex multiplication:

circular	complex
$\overline{j} = j^4$	$\overline{j} = -j^4$
$\overline{j^2} = j^3$	$\overline{j^2} = -j^3$
$\overline{j^3} = j^2$	$\overline{j^3} = -j^2$
$\overline{j^4} = j$	$\overline{j^4} = -j$

Now we look at our circular τ;

$$\tau = j + \frac{1-\sqrt{5}}{2}j^2 - \frac{1-\sqrt{5}}{2}j^3 - j^4$$

and if we calculate $\overline{\tau}$ we observe

$$\overline{\tau} = -\tau.$$

This also explains why we have opposite signs in the circular τ. The complex 5D τ reads

$$\tau = \frac{1-\sqrt{5}}{2}j + j^2 + j^3 + \frac{1-\sqrt{5}}{2}j^4$$

and if we calculate $\overline{\tau}$ we observe

$$\overline{\tau} = -\tau.$$

This also explains why we do not have opposite signs in the complex τ.
So for an exponential of the form $f(t) = e^{\tau t}$ we have

$$\overline{f(t)} = f(-t) \text{ so that } f(t)\overline{f(t)} = f(t)f(-t) = f(0) = 1$$

hence it fits the sphere-cone equation $X\overline{X} = 1$.

We close with a few pictures related to exponential circles and curves.

End of this overview of exponential circles and curves in $\mathbb{R}^3, \mathbb{C}^3$ and $\mathbb{R}^5$

Figure 2: Leonhard Euler and his exp circle.

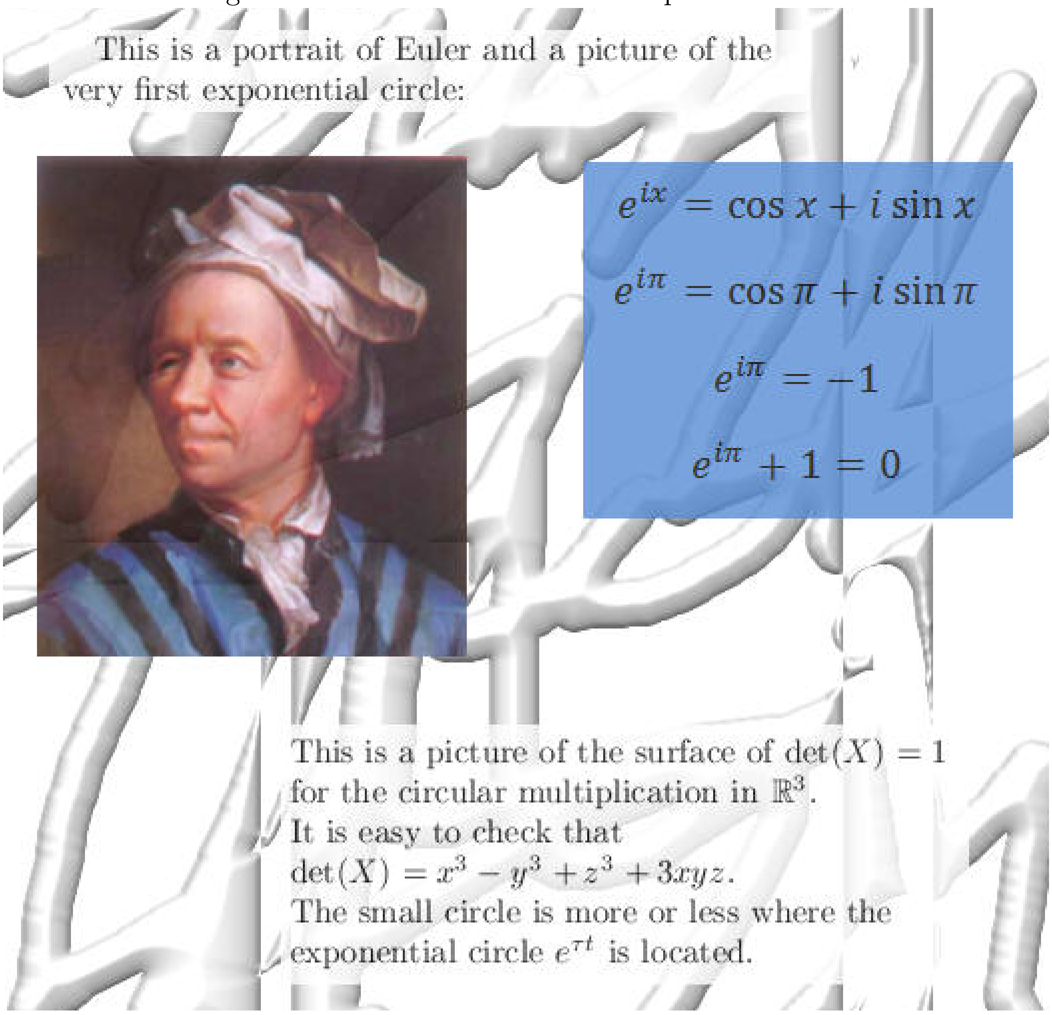


Figure 3: Surface of determinant equal to one.

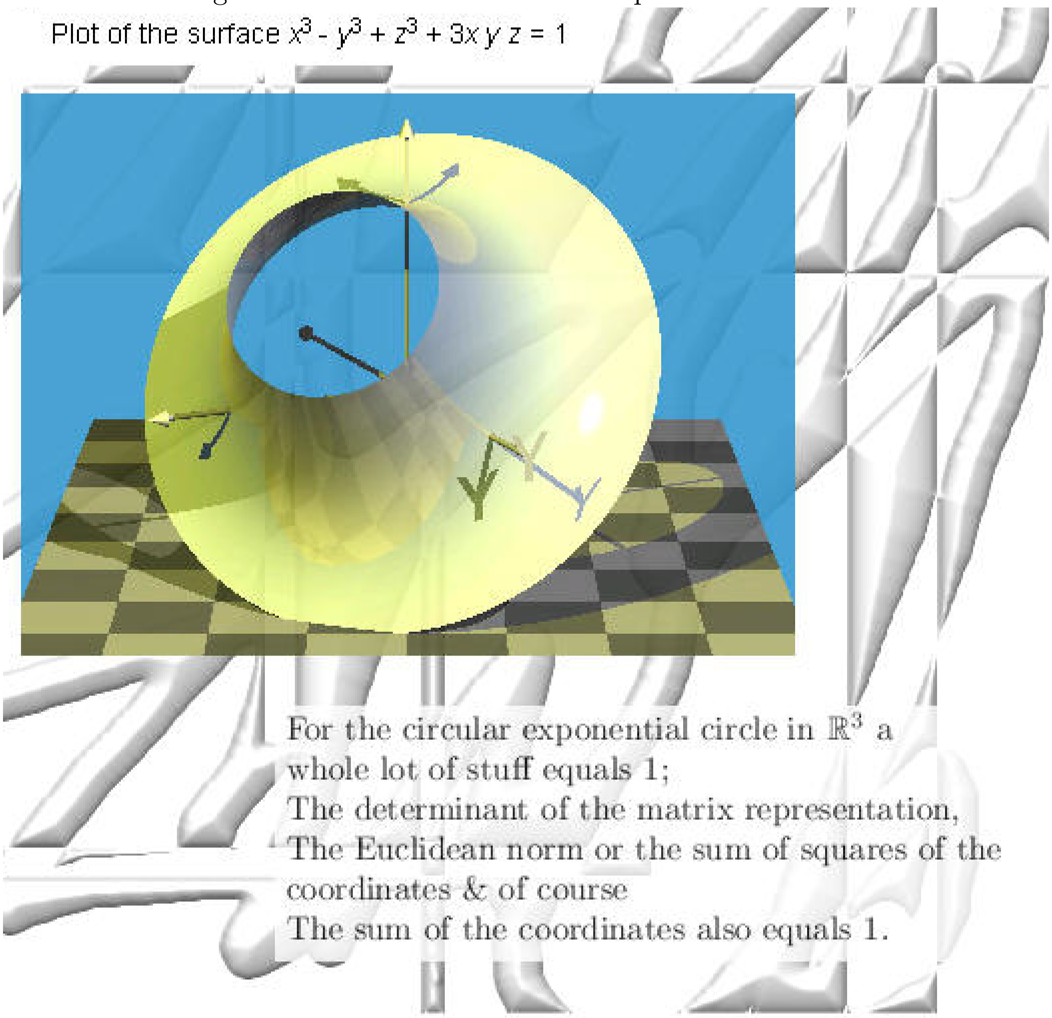
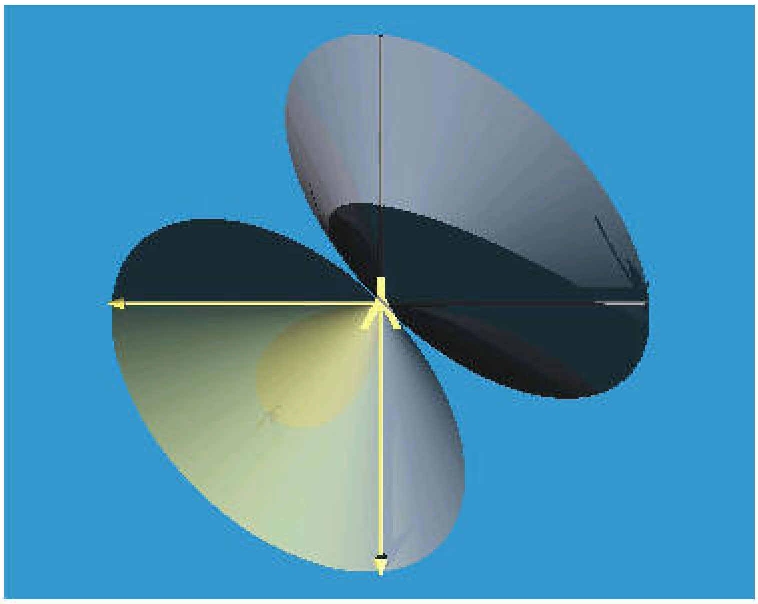


Figure 4: Surface of the cone equation equal to zero.

And last but definitely not the least of raw mathematical forces is a detail of the sphere-cone equation $X\overline{X} = 1$

Plot of the surface $y\,z + x\,z + x\,y = 0$



The upper edge should be $e^{\tau t}$ while the lower edge should be $-e^{\tau t}$.